"The Stresses and Strains in Isotropic Elastic Solid Ellipsoid in Equilibrium under Bodily Forces derivable from a Potential of the Second Degree." By C. CHREE, M.A., Fellow of King's College, Cambridge, Superintendent of Kew Observatory. Communicated by Professor W. G. ADAMS, F.R.S. Received March 2,—Read May 10, 1894. Abridged February 20, 1895.

General Formulæ.

§ 1. Let the isotropic elastic solid ellipsoid,

$$a^{-2}x^2+b^{-2}y^2+c^{-2}z^2 = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

of uniform density ρ, be acted on by bodily forces whose components $\mathrm{P}x$, $\mathrm{Q}y$, $\mathrm{R}z$ are derivable from a potential

$$\mathrm{V} = \tfrac{1}{2}(\mathrm{P}x^2+\mathrm{Q}y^2+\mathrm{R}z^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Let Π denote the determinant

$$\begin{vmatrix} 3b^4+2b^2c^2+3c^4, & c^4-\eta(b^2c^2+c^2a^2+3a^2b^2), & b^4-\eta(b^2c^2+3c^2a^2+a^2b^2) \\ c^4-\eta(b^2c^2+c^2a^2+3a^2b^2), & 3c^4+2c^2a^2+3a^4, & a^4-\eta(3b^2c^2+c^2a^2+a^2b^2) \\ b^4-\eta(b^2c^2+3c^2a^2+a^2b^2), & a^4-\eta(3b^2c^2+c^2a^2+a^2b^2), & 3a^4+2a^2b^2+3b^4 \end{vmatrix} \quad\ldots\ldots\ldots \quad (3)$$

and let its minors be Π_{11}, Π_{12}, &c., where $\Pi_{12} = \Pi_{21}$.

Then with the notation of Todhunter and Pearson's "History," η denoting Poisson's ratio, the stresses are given by

$$\left.\begin{aligned}
\widehat{xx} &= a^2\Big[(\tfrac{1}{2}\mathrm{P}\rho+\mathrm{M}+\mathrm{N})\Big(1-\frac{x^2}{a^2}\Big)-(\tfrac{1}{2}\mathrm{P}\rho+\mathrm{M}+3\mathrm{N})\frac{y^2}{b^2} \\
&\qquad -(\tfrac{1}{2}\mathrm{P}\rho+3\mathrm{M}+\mathrm{N})\frac{z^2}{c^2}\Big], \\
\widehat{yy} &= b^2\Big[-(\tfrac{1}{2}\mathrm{Q}\rho+3\mathrm{N}+\mathrm{L})\frac{x^2}{a^2}+(\tfrac{1}{2}\mathrm{Q}\rho+\mathrm{N}+\mathrm{L})\Big(1-\frac{y^2}{b^2}\Big) \\
&\qquad -(\tfrac{1}{2}\mathrm{Q}\rho+\mathrm{N}+3\mathrm{L})\frac{z^2}{c^2}\Big], \\
\widehat{zz} &= c^2\Big[-\tfrac{1}{2}(\mathrm{R}\rho+\mathrm{L}+3\mathrm{M})\frac{x^2}{a^2}-(\tfrac{1}{2}\mathrm{R}\rho+3\mathrm{L}+\mathrm{M})\frac{y^2}{b^2} \\
&\qquad +(\tfrac{1}{2}\mathrm{R}\rho+\mathrm{L}+\mathrm{M})\Big(1-\frac{z^2}{c^2}\Big)\Big], \\
\widehat{yz} &= 2\mathrm{L}yz, \qquad \widehat{zx} = 2\mathrm{M}zx, \qquad \widehat{xy} = 2\mathrm{N}xy
\end{aligned}\right\} \quad\ldots \quad (4);$$

$$\text{where } L = \frac{\rho}{2\Pi}\left[Pa^2\{\eta(b^2+c^2)\Pi_{11}+(\eta c^2-a^2)\Pi_{12}+(\eta b^2-a^2)\Pi_{13}\}\right.$$
$$+Qb^2\{(\eta c^2-b^2)\Pi_{11}+\eta(c^2+a^2)\Pi_{12}+(\eta a^2-b^2)\Pi_{13}\}$$
$$\left.+Rc^2\{(\eta b^2-c^2)\Pi_{11}+(\eta a^2-c^2)\Pi_{12}+\eta(a^2+b^2)\Pi_{13}\}\right]\ldots(5),$$

while M and N are got from L by replacing the first suffix in the Π's by 2 and by 3 respectively.

Denoting the displacements by α, β, γ, types of the 6 strains are

$$\left.\begin{aligned}
s_x \equiv \frac{d\alpha}{dx} = \frac{1}{E}\Big[&(\tfrac{1}{2}P\rho+M+N)a^2-\eta\,[(\tfrac{1}{2}Q\rho+N+L)b^2 \\
&\qquad\qquad+(\tfrac{1}{2}R\rho+L+M)c^2] \\
&-\frac{x^2}{a^2}\{(\tfrac{1}{2}P\rho+M+N)a^2-\eta\,[(\tfrac{1}{2}Q\rho+3N+L)b^2 \\
&\qquad\qquad+(\tfrac{1}{2}R\rho+L+3M)c^2]\} \\
&-\frac{y^2}{b^2}\{(\tfrac{1}{2}P\rho+M+3N)a^2-\eta\,[(\tfrac{1}{2}Q\rho+N+L)b^2 \\
&\qquad\qquad+(\tfrac{1}{2}R\rho+3L+M)c^2]\} \\
&-\frac{z^2}{c^2}\{(\tfrac{1}{2}P\rho+3M+N)a^2-\eta\,[(\tfrac{1}{2}Q\rho+N+3L)b^2 \\
&\qquad\qquad+(\tfrac{1}{2}R\rho+L+M)c^2]\}\Big], \\
\sigma_{yz} \equiv \frac{d\beta}{dz}+\frac{d\gamma}{dy} = \frac{4(1+\eta)}{E}&\,Lyz,
\end{aligned}\right\}\ldots(6);$$

where E is Young's modulus, η Poisson's ratio.

A type of the three displacements is

$$\begin{aligned}
\alpha = \frac{x}{E}\Big[&(\tfrac{1}{2}P\rho+M+N)a^2-\eta(\tfrac{1}{2}Q\rho+N+L)b^2 \\
&\qquad\qquad-\eta(\tfrac{1}{2}R\rho+L+M)c^2 \\
&-\tfrac{1}{3}\frac{x^2}{a^2}\{(\tfrac{1}{2}P\rho+M+N)a^2-\eta(\tfrac{1}{2}Q\rho+3N+L)b^2 \\
&\qquad\qquad-\eta(\tfrac{1}{2}R\rho+L+3M)c^2\} \\
&-\frac{y^2}{b^2}\{(\tfrac{1}{2}P\rho+M+3N)a^2-\eta(\tfrac{1}{2}Q\rho+N+L)b^2 \\
&\qquad\qquad-\eta(\tfrac{1}{2}R\rho+3L+M)c^2\} \\
&-\frac{z^2}{c^2}\{(\tfrac{1}{2}P\rho+3M+N)a^2-\eta(\tfrac{1}{2}Q\rho+N+3L)b^2 \\
&\qquad\qquad-\eta(\tfrac{1}{2}R\rho+L+M)c^2\}\Big]\ldots(7).
\end{aligned}$$

The other strains and displacements can be written down from symmetry.

For the elastic increment δa, in a principal semi-axis a, we have

$$\delta a/a = \frac{2}{3\mathrm{E}}\left[(\tfrac{1}{2}\mathrm{P}\rho+\mathrm{M}+\mathrm{N})\,a^2-\eta\,\{\tfrac{1}{2}\rho\,(\mathrm{Q}b^2+\mathrm{R}c^2)\right.$$
$$\left.+\mathrm{L}\,(b^2+c^2)\}\right] \;\ldots\; (8).$$

§ 2. Let $t_x\ldots.$ denote the component stresses across the tangent plane at x, y, z to a quadric concentric with, and similar and similarly situated to the bounding surface (1), for whose equation we take

$$a^{-2}x^2+b^{-2}y^2+c^{-2}z^2=\lambda \;\ldots\ldots\ldots\ldots\ldots\; (9),$$

then we easily find

$$t_x\Big/\left\{\frac{p_\lambda x}{\lambda a^2}(\tfrac{1}{2}\mathrm{P}\rho+\mathrm{M}+\mathrm{N})\,a^2\right\}=t_y\Big/\left\{\frac{p_\lambda y}{\lambda b^2}\,(\tfrac{1}{2}\mathrm{Q}\rho+\mathrm{N}+\mathrm{L})\,b^2\right\}$$
$$=t_z\Big/\left\{\frac{p_\lambda z}{\lambda c^2}(\tfrac{1}{2}\mathrm{R}\rho+\mathrm{L}+\mathrm{M})c^2\right\}=1-\lambda \;\ldots\ldots\ldots\; (10),$$

where p_λ is the perpendicular from the centre on the tangent plane. Thus the resultant stresses across parallel tangent planes to the system (9) at the points of contact are all parallel, and their intensity varies as $1-\lambda$.

§ 3. If the ellipsoid be rotating with uniform angular velocity ω about the axis $2a$, we have

$$\mathrm{P}=0,\;\; \mathrm{Q}=\mathrm{R}=\omega^2;$$

while if it be gravitating, the force between unit masses at unit distance being taken as unity,

$$\mathrm{P}=2\pi\rho abc\int_0^\infty \frac{du}{\sqrt{\{(a^2+u)^3(b^2+u)(c^2+u)\}}} \;\ldots\ldots\ldots\; (11)^*$$

with symmetrical expressions for Q and R. If the ellipsoid be gravitating, and at the same time rotating about a principal axis, we have only to add the respective values of P, Q, R. Substituting the values of P, Q, R in the expressions for L, M, N, and inserting the consequent values of L, M, N in the formulæ (4), (6), (7), we have the complete values of the stresses, strains, and displacements.

Gravitating nearly Spherical Ellipsoid.

§ 4. Denoting by μ the force between two unit masses at unit distance, we may take

$$\mathrm{P}=-\frac{4}{3}\pi\mu\rho\left(1-\frac{2a^2-b^2-c^2}{5a^2}\right)^*,$$

with symmetrical expressions for Q and R.

We thence find

* Thomson and Tait's 'Natural Philosophy' vol. 1, Part II, p. 47.

$$\widehat{xx} = \frac{2\pi\mu\rho^2 a^2}{15(1-\eta)(7+5\eta)} \Big[-\left(1-\frac{x^2}{a^2}\right) \Big\{ (7+5\eta)(3-\eta) - \frac{2a^2-b^2-c^2}{a^2}(2-\eta)(3+\eta) \Big\}$$

$$+\frac{y^2}{b^2}\Big\{ (7+5\eta)(1+3\eta) - \frac{a^2-b^2}{a^2}(8-3\eta-9\eta^2) - \frac{a^2-c^2}{a^2}(2+3\eta+15\eta^2) \Big\}$$

$$+\frac{z^2}{c^2}\Big\{ (7+5\eta)(1+3\eta) - \frac{a^2-b^2}{a^2}(2+3\eta+15\eta^2) - \frac{a^2-c^2}{a^2}(8-3\eta-9\eta^2) \Big\} \Big] \quad \ldots\ldots (12),$$

$$\widehat{yz} = \frac{4\pi\mu\rho^2 yz}{15(1-\eta)(7+5\eta)} \Big\{ (7+5\eta)(1-2\eta) + \frac{2a^2-b^2-c^2}{a^2}(1-\eta-4\eta^2) \Big\} \quad \ldots\ldots (13).$$

The other stresses may be written down from symmetry.

With the notation of § 2,

$$t_x = -p_\lambda \frac{1-\lambda}{\lambda}\,\frac{2\pi\mu\rho^2 x(3-\eta)}{15(1-\eta)} \Big\{ 1 - \frac{2a^2-b^2-c^2}{a^2}\,\frac{(2-\eta)(3+\eta)}{(7+5\eta)(3-\eta)} \Big\} \quad \ldots (14).$$

It will be noticed that

$$p_\lambda \frac{1-\lambda}{\lambda} x = px'(1-r^2/r'^2),$$

where $\qquad r = \sqrt{(x^2+y^2+z^2)}, \; r' = \sqrt{(x'^2+y'^2+z'^2)},$

x', y', z' being the coordinates of the point where the radius vector r produced cuts the surface of the ellipsoid, and p being the perpendicular from the centre on the tangent plane at x', y', z'. Near the surface we may put

$$1-r^2/r'^2 = 2(1-r/r'),$$

and so conclude that t_x and the other stress components *across* the tangent planes to (q) vary approximately as the distance from the surface.

As the stresses at the surface itself are of special interest in the event of any application to the earth, I shall briefly consider them for a spheroid in which $b = a$. The principal stresses are $\widehat{nn}$, $\widehat{tt}$, $\widehat{\phi\phi}$, directed respectively along the normal, the tangent to the meridian, and the perpendicular to the meridian. Using cylindrical coordinates, r, ϕ, z where

$$r = \sqrt{(x^2+y^2)}, \quad \phi = \tan^{-1}y/x,$$

we get

$$\left.\begin{aligned}\widehat{nn} &= 0,\\ \widehat{tt} &= \frac{-4\pi\mu\rho^2a^2}{15(1-\eta)}\left\{\frac{c^2}{p^2}(1-2\eta)+\frac{a^2-c^2}{a^2}\,\frac{1-\eta-4\eta^2}{7+5\eta}\right\},\\ \widehat{\phi\phi} &= \frac{-4\pi\mu\rho^2a^2}{15(1-\eta)}\left\{1-2\eta+\frac{a^2-c^2}{a^2}\left(1-\frac{3r^2}{a^2}\right)\frac{1-\eta-4\eta^2}{7+5\eta}\right\}\end{aligned}\right\}\ \ldots\ldots\ (15)$$

where p is the perpendicular from the centre on the tangent plane.

On the "stress-difference" theory of rupture an objection to the application to the earth of the results obtained by applying the elastic solid theory to a perfect sphere, is that the surface values of $\widehat{nn}-\widehat{\phi\phi}$ and $\widehat{nn}-\widehat{tt}$ would, for ordinary values of η, be simply enormous.* This objection, however, ceases to hold when the earth is treated as incompressible and truly spherical, because $\widehat{tt}$ and $\widehat{\phi\phi}$ then vanish, as well as $\widehat{nn}$. It is thus important to know what happens in the case of an incompressible material when the surface is slightly spheroidal. To do so, put $\eta=\frac{1}{2}$ in (15), and we find

$$\left.\begin{aligned}\widehat{nn} &= 0,\\ \widehat{tt} &= 8\pi\mu\rho^2(a^2-c^2)/285,\\ \widehat{\phi\phi} &= 8\pi\mu\rho^2(a^2-c^2)(1-3r^2/a^2)/285\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ (16).$$

Over the surface the maximum stress-difference, $\bar{S}$, is the equatorial value of $\widehat{tt}-\widehat{\phi\phi}$, and is given by

$$\bar{S} = 8\pi\mu\rho^2(a^2-c^2)/95\ \ldots\ldots\ldots\ldots\ldots\ldots\ (17).$$

Substituting for ρ, a, c values suitable to the case of a homogeneous "earth," we find that approximately

$$\bar{S} = 9{\cdot}4 \text{ tons weight per square inch}\ \ldots\ldots\ldots\ldots\ (18).$$

This is large enough to show that even if the earth be supposed incompressible, the consequences of its mutual gravitation cannot safely be ignored.

The strains and displacements in the general case of gravitation in a nearly spherical ellipsoid may easily be deduced from (6) and (7). From the expressions for the changes in the lengths of the semi-axes we get

$$\frac{\delta a}{a} = -\frac{4}{15}\,\frac{\pi\mu\rho^2a^2}{E}\left\{1-2\eta-\frac{2a^2-b^2-c^2}{a^2}\,\frac{2(1-3\eta-3\eta^2)}{7+5\eta}\right\}\ \ldots\ (19),$$

$$\frac{\delta b}{b}-\frac{\delta c}{c} = -\frac{4}{15}\,\pi\mu\rho^2(b^2-c^2)\frac{(1+\eta)(1+8\eta)}{E(7+5\eta)}\ \ldots\ldots\ (20).$$

* See 'Phil. Mag.,' Sept., 1891, p. 247.

The principal axes all shorten in any material which is not very nearly incompressible.

For absolutely incompressible material,

$$\frac{\delta a}{a} = -\frac{4}{57}\frac{\pi\mu\rho^2 a^2}{\mathrm{E}}\frac{2a^2-b^2-c^2}{a^2} \quad \ldots\ldots\ldots\ldots (21);$$

thus a principal axis shortens or lengthens according as the square on it is greater or less than the arithmetic mean of the squares on the principal axes. In a spheroid $b = a$, we find from (21),

$$-\frac{\delta a}{a} = \tfrac{1}{2}\frac{\delta c}{c} = \frac{4}{57}\frac{\pi\mu\rho^2(a^2-c^2)}{\mathrm{E}} = \frac{5}{6}\frac{\overline{\mathrm{S}}}{\mathrm{E}} \quad \ldots\ldots\ldots\ldots (22),$$

where $\overline{\mathrm{S}}$ is given by (17).

Taking the numerical value (18) for $\overline{\mathrm{S}}$, and for E the high value 20×10^8 grammes weight per square centimetre, we should get from (22) for a spheroid the shape and size of the earth, a shortening of some 5 miles in equatorial diameters and a lengthening of some 10 miles in the polar diameter relative to what these lengths would have been in the absence of gravitation.

Rotating nearly Spherical Ellipsoid.

§ 5. Suppose next that the nearly spherical ellipsoid rotates with uniform angular velocity ω about $2a$. The values of $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$, and $\widehat{yz}$ are as follows:—

$$\begin{aligned}\widehat{xx} = \frac{-\omega^2\rho a^2}{5(1-\eta)(7+5\eta)^2}\Bigg[&\left(1-\frac{x^2}{a^2}\right)\Big\{(7+5\eta)(3-6\eta-5\eta^2) \\ &\qquad -\frac{2a^2-b^2-c^2}{a^2}\,2(1+\eta)(6-5\eta-5\eta^2)\Big\} \\ &-\frac{y^2}{b^2}\Big\{2(7+5\eta)(3-6\eta-5\eta^2)-\frac{a^2-b^2}{a^2}(39+16\eta-29\eta^2-10\eta^3) \\ &\qquad -\frac{a^2-c^2}{a^2}(9-8\eta-51\eta^2-30\eta^3)\Big\} \\ &-\frac{z^2}{c^2}\Big\{2(7+5\eta)(3-6\eta-5\eta^2)-\frac{a^2-b^2}{a^2}(9-8\eta-51\eta^2-30\eta^3) \\ &\qquad -\frac{a^2-c^2}{a^2}(39+16\eta-29\eta^2-10\eta^3)\Big\}\Bigg] \quad \ldots\ldots (23),\end{aligned}$$

$$\widehat{yy} = \frac{w^2 \rho b^2}{5(1-\eta)(7+5\eta)^2}\Big[-\frac{x^2}{a^2}\Big\{(7+5\eta)(9+7\eta)$$

$$+\frac{a^2-b^2}{a^2}(39+23\eta-3\eta^2+5\eta^3)-\frac{a^2-c^2}{a^2}(6+13\eta+36\eta^2+25\eta^3)\Big\}$$

$$+\left(1-\frac{y^2}{b^2}\right)\Big\{(7+5\eta)(12+\eta-5\eta^2)+\frac{a^2-b^2}{a^2}(12+9\eta+6\eta^2+5\eta^3)$$

$$-\frac{a^2-c^2}{a^2}(1+\eta)(3+5\eta^2)\Big\}$$

$$-\frac{z^2}{c^2}\Big\{(7+5\eta)(4+7\eta+5\eta^2)+\frac{a^2-b^2}{a^2}(9+13\eta+27\eta^2+15\eta^3)$$

$$-\frac{a^2-c^2}{a^2}(6-\eta-16\eta^2-6k^3)\Big\}\Big] \quad \ldots\ldots \quad (24),$$

$$\widehat{xy} = \frac{-w^2 \rho xy}{5(1-\eta)(7+5\eta)^2}\Big[(7+5\eta)(3-6\eta-5\eta^2)$$

$$-\frac{a^2-b^2}{a^2}(27+14\eta-9\eta^2)+\frac{a^2-c^2}{a^2}(3+10\eta+31\eta^2+20\eta^3)\Big]$$

$$\ldots\ldots \quad (25),$$

$$\widehat{yz} = \frac{-w^2 \rho yz}{5(1-\eta)(7+5\eta)^2}\Big[2(7+5\eta)(4-3\eta-5\eta^2)$$

$$+\frac{2a^2-b^2-c^2}{a^2}(3-4\eta-21\eta^2-10\eta^3)\Big] \quad \ldots\ldots \quad (26).$$

$\widehat{zz}$ may be obtained from $\widehat{yy}$, and $\widehat{zx}$ from $\widehat{xy}$ by interchanging y with z and b with c.

The reduction in length along the axis of rotation is given by

$$(-\delta a/a) = \frac{2w^2 \rho a^2}{5\mathrm{E}(7+5\eta)}\Big[1+8\eta+5\eta^2$$

$$-\frac{2a^2-b^2-c^2}{a^2}\,\frac{24+223\eta+282\eta^2+95\eta^3}{6(7+5\eta)}\Big] \quad \ldots\ldots \quad (27).$$

It is thus greater or less than in a sphere of radius a, according as a^2 is less or greater than the arithmetic mean of the squares on the principal semi-axes. The tendency to shorten in the axis of rotation increases as the perpendicular diameters lengthen.

For the increment in a perpendicular semi-axis b, we get

$$\delta b/b = \frac{w^2 \rho a^2}{5\mathrm{E}(7+5\eta)}\Big[8-\eta-5\eta^2+\frac{b^2+c^2-2a^2}{a^2}\,\frac{25+16\eta-15\eta^2-10\eta^3}{7+5\eta}$$

$$+\frac{b^2-c^2}{a^2}\,\frac{(1+\eta)(69+137\eta+70\eta^2)}{3(7+5\eta)}\Big] \quad \ldots\ldots \quad (28).$$

From this we see that the "extension per unit of length"* is greatest in the longer of the two principal diameters perpendicular to the axis of rotation.

Very Flat Ellipsoid.

§ 6. Supposing $2c$ the short axis, we find, neglecting higher powers of c/a and b/a,

$$\widehat{xx} = \tfrac{1}{3}\rho\left[\mathrm{P}a^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\mathrm{R}c^2\frac{\eta}{2(1-\eta)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-3\frac{z^2}{c^2}\right)\right.$$
$$\left.+\frac{\mathrm{P}a^2(\eta b^2-a^2)+\mathrm{Q}b^2(\eta a^2-b^2)+\mathrm{R}c^2\eta(a^2+b^2)}{4a^4+3a^2b^2+4b^4+\eta a^2b^2}a^2\left(1-\frac{x^2}{a^2}-\frac{4y^2}{b^2}\right)\right] \quad \ldots\ldots (29),$$

$$\widehat{zz} = \tfrac{1}{6}\rho c^2\left[-\mathrm{P}\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-\mathrm{Q}\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}-\frac{z^2}{c^2}\right)\right.$$
$$+3\mathrm{R}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)$$
$$\left.-2\frac{\mathrm{P}a^2(\eta b^2-a^2)+\mathrm{Q}b^2(\eta a^2-b^2)}{4a^4+3a^2b^2+4b^4+\eta a^2b^2}\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}-\frac{z^2}{c^2}\right)\right] \quad \ldots (30),$$

$$\widehat{xy} = \rho xy\,\frac{\{\mathrm{P}a^2(\eta b^2-a^2)+\mathrm{Q}b^2(\eta a^2-b^2)+\mathrm{R}c^2\eta(a^2+b^2)\}}{4a^4+3a^2b^2+4b^4+\eta a^2b^2} \quad \ldots\ldots\ldots (31),$$

$$\widehat{zx} = \tfrac{1}{3}\rho xz\left\{-\mathrm{P}+\frac{\eta}{1-\eta}\frac{\mathrm{R}c^2}{a^2}\right.$$
$$\left.-\frac{\mathrm{P}a^2(\eta b^2-a^2)+\mathrm{Q}b^2(\eta a^2-b^2)+\mathrm{R}c^2\eta(a^2+b^2)}{4a^4+3a^2b^2+4b^4+\eta a^2b^2}\right\} \quad \ldots.(32).$$

$\widehat{yy}$ is got from $\widehat{xx}$, and $\widehat{yz}$ from $\widehat{zx}$ by interchanging P with Q and a with b.

The forces P, Q, R may occur independently, so the most important term in the coefficient of each has been retained.

When P and Q alone exist, or the forces are perpendicular to the small dimension, $\widehat{zx}$ and $\widehat{yz}$ bear to $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$ ratios of the order $c : a$, while $\widehat{zz}$ bears a ratio of the order $c^2 : a^2$. Thus for a first approximation the three stresses $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$ alone need be retained. To a first approximation $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$ are constant along any line parallel to the small dimension. These conclusions

* Used here and subsequently in sense of total change in length of axis divided by original length, whether strain be uniform or not along the axis.

are so far in harmony with the assumptions usually made in theories of thin plates. It should be noticed, however, that $d\widehat{zx}/dz$ is of the same order of magnitude as $d\widehat{xx}/dx$, and so of equal importance in the first body-stress equation.

When R alone exists, or the forces are parallel to the small dimension, $\widehat{zz}$ at points in the interior is of the same order of magnitude as $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$.

If there be bodily forces both parallel and perpendicular to the small dimension, and the two sets be of the same order of magnitude, then to a first approximation the stresses due to the former set of forces may be neglected.

Gravitating very Oblate Spheroid.

§7. Taking μ as before for the gravitation constant, we have as a first approximation *

$$P = Q = -\pi^2\mu\rho c/a, \qquad R = -4\pi\mu\rho \quad \ldots\ldots\ldots \quad (33).$$

As there is symmetry round the axis of z, we employ cylindrical coordinates r, ϕ, z. The notation $\widehat{rr}$, $\widehat{\phi\phi}$, ... for the stresses explains itself. The displacements are u along r, and γ parallel to z. The strains are

$$\left.\begin{aligned} s_r &= du/dr, & s_\phi &= u/r, & s_z &= d\gamma/dz, \\ \sigma_{r\phi} &= 0, & \sigma_{\phi z} &= 0, & \sigma_{zr} &= du/dz + d\gamma/dr \end{aligned}\right\} \ldots \quad (34).$$

For the values of the stresses we find

$$\left.\begin{aligned} \widehat{rr} &= -\pi^2\mu\rho^2 ac\,\frac{3+\eta}{11+\eta}\left(1-\frac{r^2}{a^2}\right), \\ \widehat{\phi\phi} &= -\frac{\pi^2\mu\rho^2 ac}{11+\eta}\left\{3+\eta-(1+3\eta)\frac{r^2}{a^2}\right\}, \\ \widehat{zz} &= -2\pi\mu\rho^2 c^2\,(1-r^2/a^2-z^2/c^2), \\ \widehat{zr} &= \pi^2\mu\rho^2\,\frac{3+\eta}{11+\eta}\,\frac{c}{a}\,zr, \\ \widehat{r\phi} &= \widehat{\phi z} = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (35).$$

The strains which differ from zero are

* Thomson and Tait's 'Natural Philosophy,' vol. 1, Part II, Art. 527.

$$\left.\begin{aligned} s_r &= -\frac{\pi^2\mu\rho^2 ac(1-\eta)}{E(11+\eta)}\left\{3+\eta-3(1+\eta)\frac{r^2}{a^2}\right\}, \\ s_\phi &= -\frac{\pi^2\mu\rho^2 ac(1-\eta)}{E(11+\eta)}\left\{3+\eta-(1+\eta)\frac{r^2}{a^2}\right\}, \\ s_z &= \frac{2\pi^2\mu\rho^2 ac\eta}{E(11+\eta)}\left\{3+\eta-2(1+\eta)\frac{r^2}{a^2}\right\}, \\ \sigma_{zr} &= 2\pi^2\mu\rho^2\frac{c}{a}zr\frac{(1+\eta)(3+\eta)}{E(11+\eta)} \end{aligned}\right\} \ldots\ldots (36).$$

The displacements and dilatation are

$$\left.\begin{aligned} u &= -\frac{\pi^2\mu\rho^2 acr(1-\eta)}{E(11+\eta)}\left\{3+\eta-(1+\eta)\frac{r^2}{a^2}\right\}, \\ \gamma &= \frac{2\pi^2\mu\rho^2 acz\eta}{E(11+\eta)}\left\{3+\eta-2(1+\eta)\frac{r^2}{a^2}\right\}, \\ \Delta &= -\frac{2\pi^2\mu\rho^2 ac(1-2\eta)}{E(11+\eta)}\left\{3+\eta-2(1+\eta)\frac{r^2}{a^2}\right\} \end{aligned}\right\} \ldots (37).$$

Strictly, $\widehat{zz}$, $\widehat{zr}$, σ_{zr} and γ, being of the order of terms omitted in $\widehat{rr}$, s_r, &c., should be neglected; but the record of their first approximations may prove useful.

$\widehat{rr}$, $\widehat{\phi\phi}$ and $\widehat{zz}$ are *pressures* at every interior point; s_ϕ is everywhere a compression, and s_z an extension, while s_r is a compression inside and an extension outside the cylinder

$$r = r_1 = a\sqrt{\{\tfrac{1}{3}(3+\eta)/(1+\eta)\}} \ldots\ldots\ldots\ldots (38).$$

The displacement u is everywhere towards the axis of the spheroid, and γ away from the central plane, $z = 0$. Every element is reduced in volume.

Flat Ellipsoid Rotating about the Short Axis 2c.

§ 8. Here

$$P = Q = \omega^2, \qquad R = 0.$$

Putting for shortness

$$4a^4+3a^2b^2+4b^4+\eta a^2b^2 = \Pi' \ldots\ldots\ldots\ldots (39),$$

we find

$$\left.\begin{aligned}
\widehat{xx} &= \omega^2\rho a^2\left[\{a^4+a^2b^2(1+\eta)+b^4\}(1-x^2/a^2)\right.\\
&\qquad\qquad\qquad\qquad\left.-(1+3\eta)\,a^2y^2\right]\div\Pi',\\
\widehat{yy} &= \omega^2\rho b^2\left[-(1+3\eta)\,b^2x^2\right.\\
&\qquad\left.+\{a^4+a^2b^2(1+\eta)+b^4\}(1-y^2/b^2)\right]\div\Pi',\\
\widehat{zz} &= -\omega^2\rho c^2\{a^4+a^2b^2(1+\eta)+b^4\}\\
&\qquad\times\{1-2x^2/a^2-2y^2/b^2-z^2/c^2\}\div\Pi',\\
\widehat{xy} &= -\omega^2\rho\,(a^4-2\eta a^2b^2+b^4)\,xy\div\Pi',\\
\widehat{yz}/yz &= \widehat{zx}/zx = -\omega^2\rho\{a^4+a^2b^2(1+\eta)+b^4\}\div\Pi'
\end{aligned}\right\}\ldots (40).$$

$\widehat{zz}$, it should be noticed, is negligible compared to $\widehat{yz}$ and $\widehat{zx}$, and these in turn negligible compared to $\widehat{xx}$, $\widehat{yy}$ and $\widehat{xy}$. The values of $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$ bear to the corresponding first approximations in the case of an elliptic disc* of semi-axes a and b, rotating about its cylindrical axis with the same angular velocity ω, the common ratio

$$3a^4+2a^2b^2+3b^4 : 4a^4+(3+\eta)\,a^2b^2+4b^4 \ldots\ldots\ldots (41).$$

This ratio also applies to the first approximations to Δ, s_x, s_y, s_z, and σ_{xy}; while the other strains are in both cases negligible compared to these. It applies further to $\delta a/a$ and $\delta b/b$. Thus the great majority of the results worked out for the rotating disc can be simply modified so as to apply to the flat ellipsoid by means of the following table which proceeds to three places of decimals.

Table I.—Values of $(3a^4+2a^2b^2+3b^4)\div\{4a^4+(3+\eta)\,a^2b^2+4b^4\}$.

η	b/a 0.	0·2	0·4	0·6	0·8	1·0
0	0·75	0·748	0·741	0·734	0·729	0·$\dot{7}\dot{2}$
0·25	0·75	0·746	0·735	0·722	0·714	0·71
0·5	0·75	0·744	0·729	0·711	0·699	0·696

On either the "greatest strain" or the "maximum stress-difference" theory of rupture, the limiting angular velocity in the flat ellipsoid

* 'Phil. Mag.,' July, 1892, pp. 70—100; see results (17), (18), (19), p. 75.

bears to that in the thin disc, of the same material and axes, a ratio which is the square root of the reciprocal of (41).

If the limiting angular velocities be ω for the ellipsoid, ω' for the disc, then the approximate values of ω/ω' when $\eta = \cdot 25$, are as follows:—

Table II.

$b/a =$	0	0·2	0·4	0·6	0·8	1·0
$\omega/\omega' =$	1·155	1·158	1·167	1·177	1·184	1·186

In the case of the thin disc many very simple results were found for the radial strains; these apply without modification to the strain s_r in the flat ellipsoid.

Flat Ellipsoid rotating about one of its longer axes 2a.

§ 9. Putting $P = 0, \quad Q = R = \omega^2$,
and employing Π' as in (39), we find

$$\left.\begin{aligned}
\widehat{xx} &= \frac{1}{3\Pi'}\,\omega^2\rho a^2 b^2\,(\eta a^2 - b^2)\left(1 - \frac{x^2}{a^2} - \frac{4y^2}{b^2}\right),\\
\widehat{yy} &= \tfrac{1}{3}\,\omega^2\rho b^2\left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{b^2(\eta a^2 - b^2)}{\Pi'}\left(1 - \frac{4x^2}{a^2} - \frac{y^2}{b^2}\right)\right\},\\
\widehat{zz} &= \tfrac{1}{3}\,\omega^2\rho c^2\left\{1 - \frac{x^2}{a^2} - \frac{z^2}{c^2} - \frac{b^2(\eta a^2 - b^2)}{\Pi'}\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2} - \frac{z^2}{c^2}\right)\right\},\\
\widehat{yz} &= -\tfrac{1}{3}\,\omega^2\rho yz\left\{1 + \frac{b^2(\eta a^2 - b^2)}{\Pi'}\right\},\\
\widehat{zx} &= -\tfrac{1}{3}\,\frac{\omega^2\rho b^2}{\Pi'}\,(\eta a^2 - b^2)\,zx,\\
\widehat{xy} &= \frac{\omega^2\rho b^2}{\Pi'}\,(\eta a^2 - b^2)\,xy
\end{aligned}\right\} \ldots (42).$$

As b/a passes through the critical value $\sqrt{\eta}$, the stresses $\widehat{xx}$, $\widehat{zx}$, and $\widehat{xy}$ vanish and change sign; and at the critical value, to a first approximation, the only stress is

$$\widehat{yy} = \tfrac{1}{3}\omega^2\rho b^2(1 - x^2/a^2 - y^2/b^2) \ldots\ldots\ldots\ldots\ldots (43)$$

The stretches answering to (42) are

$$
\left.\begin{aligned}
&= -\tfrac{1}{3}\frac{\omega^2\rho b^2}{\mathrm{E}}\left[\left\{\eta+\frac{(a^2-\eta b^2)(b^2-\eta a^2)}{\Pi'}\right\}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\qquad\left.+3\frac{(b^2-\eta a^2)}{\Pi'}\left(\eta\frac{b^2}{a^2}x^2-\frac{a^2}{b^2}y^2\right)\right],\\
s_y &= \tfrac{1}{3}\frac{\omega^2\rho b^2}{\mathrm{E}}\left[\left\{1-\frac{(b^2-\eta a^2)^2}{\Pi'}\right\}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\qquad\left.+\frac{3(b^2-\eta a^2)}{\Pi'}\left(\frac{b^2}{a^2}x^2-\eta\frac{a^2}{b^2}y^2\right)\right],\\
s_z &= -\tfrac{1}{3}\frac{\omega^2\rho b^2\eta}{\mathrm{E}}\left[\left\{1+\frac{(\eta a^2-b^2)(a^2+b^2)}{\Pi'}\right\}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\qquad\left.+\frac{3(b^2-\eta a^2)}{\Pi'}\left(\frac{b^2}{a^2}x^2+\frac{a^2}{b^2}y^2\right)\right]
\end{aligned}\right\} \ldots (44).
$$

For the increments in the semi-axes we have

$$
\left.\begin{aligned}
\frac{\delta a}{a} &= -\tfrac{1}{9}\frac{\omega^2\rho b^2}{\mathrm{E}}\{2\eta+(b^2-\eta a^2)(2a^2+\eta b^2)/\Pi'\},\\
\frac{\delta b}{b} &= \tfrac{1}{9}\frac{\omega^2\rho b^2}{\mathrm{E}}\{2-(b^2-\eta a^2)(\eta a^2+2b^2)/\Pi'\},\\
\frac{\delta c}{c} &= -\tfrac{1}{3}\frac{\omega^2\rho b^2\eta}{\mathrm{E}}\{1-(b^2-\eta a^2)(a^2+b^2)/\Pi'\}
\end{aligned}\right\} \ldots\ldots (45).
$$

It is easy to prove $\delta b/b$ positive, and $\delta c/c$ negative, for all values of η and b/a; thus, of the axes perpendicular to the axis of rotation, the longer lengthens and the shorter shortens. The axis $2a$ of rotation is always reduced.

When the flat ellipsoid is spheroidal $(-\,\delta c/c)$ is greater or less than $(-\,\delta a/a)$, according as η is greater or less than $\cdot25$.

Very Elongated Ellipsoid.

§ 10. Treating a/c and b/c as both very small, we find

$$
\begin{aligned}
&\widehat{xx}\,.\,4(1-\eta^2)(3a^4+2a^2b^2+3b^4)\\
&= 2\mathrm{P}\rho a^2(1+\eta)\left[\{(1-\eta)(2a^4+2a^2b^2+3b^4)+\eta a^2b^2\}\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)\right.\\
&\qquad\qquad\left.-\{2a^2+3b^2+\eta(a^2-3b^2)\}y^2\right]\\
&\quad+2\mathrm{Q}\rho b^2a^2(1+\eta)\{\eta(a^2+b^2)-b^2\}\left\{1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}-\frac{z^2}{c^2}\right\}
\end{aligned}
$$

$$-\mathrm{R}\rho a^2\Big[\{(1-\eta^2)(a^4+a^2b^2+b^4)-\eta(1+3\eta)a^2b^2\}\left(1-\frac{x^2}{a^2}\right)$$

$$-a^2b^2(1+2\eta)(1-5\eta)\frac{y^2}{b^2}$$

$$-\{(1-\eta^2)(4a^4+3a^2b^2+4b^4)-\eta(1+3\eta)a^2b^2\}\frac{z^2}{c^2}\Big] \quad \ldots (46),$$

$$\widehat{zz}\,(1-\eta^2)\left[4(3a^4+2a^2b^2+3b^4)+\frac{a^2+b^2}{c^2}\{(3+\eta)(3a^4+2a^2b^2+3b^4)+8\eta a^2b^2\}\right]$$

$$=\mathrm{P}\rho a^2\eta\Big[-2\{3a^4+2a^2b^2+3b^4-(a^2-\eta b^2)(a^2+3b^2)-2\eta^2a^2(a^2-b^2)\}\frac{x^2}{a^2}$$

$$-2\{3a^4+2a^2b^2+3b^4-(a^2-\eta b^2)(3a^2+b^2)+2\eta^2b^2(a^2-b^2)\}\frac{y^2}{b^2}$$

$$+\{3a^4+2a^2b^2+3b^4-2(a^2-\eta b^2)(a^2+b^2)-\eta^2(a^2-b^2)^2\}\left(1-\frac{z^2}{c^2}\right)\Big]$$

$$+\mathrm{Q}\rho b^2\eta\Big[-2\{3a^4+2a^2b^2+3b^4-(b^2-\eta a^2)(a^2+3b^2)-2\eta^2a^2(a^2-b^2)\}\frac{x^2}{a^2}$$

$$-2\{3a^4+2a^2b^2+3b^4-(b^2-\eta a^2)(3a^2+b^2)+2\eta^2b^2(a^2-b^2)\}\frac{y^2}{b^2}$$

$$+\{3a^4+2a^2b^2+3b^4-2(b^2-\eta a^2)(a^2+b^2)-\eta^2(a^2-b^2)^2\}\left(1-\frac{z^2}{c^2}\right)\Big]$$

$$+\mathrm{R}\rho c^2\Big[-\{3a^4+2a^2b^2+3b^4+\eta(5a^4+3a^2b^2+4b^4)-\eta^2a^2(3a^2+b^2)-5\eta^3(a^4-b^4)\}\frac{x^2}{c^2}$$

$$-\{3a^4+2a^2b^2+3b^4+\eta(4a^4+3a^2b^2+5b^4)-\eta^2b^2(a^2+3b^2)+5\eta^3(a^4-b^4)\}\frac{y^2}{c^2}$$

$$+\Big\{(1-\eta^2)(3a^4+2a^2b^2+3b^4)+\frac{a^2+b^2}{c^2}\Big(3a^4+2a^2b^2+3b^4+\eta(2a^4+3a^2b^2+2b^4)$$

$$-\eta^2(3a^4+a^2b^2+3b^4)-2\eta^3(a^4+b^4)\Big)\Big\}\left(1-\frac{z^2}{c^2}\right)\Big] \quad \ldots\ldots (47),$$

$$\widehat{xy}\,.\,4(1-\eta^2)(3a^4+2a^2b^2+3b^4)$$

$$=\rho xy[4(1+\eta)\mathrm{P}a^2\{\eta(a^2+b^2)-a^2\}+4(1+\eta)\mathrm{Q}b^2\{\eta(a^2+b^2)-b^2\}$$

$$+\mathrm{R}\{(1-\eta^2)(a^4+b^4)+2\eta(1+3\eta)a^2b^2\}] \quad \ldots (48),$$

$$\widehat{zx}\,(1-\eta^2)\left[4(3a^4+2a^2b^2+3b^4)+\frac{a^2+b^2}{c^2}\{(3+\eta)(3a^4+2a^2b^2+3b^4)+8\eta a^2b^2\}\right]$$

$$=\rho zx\Big[\mathrm{P}\frac{a^2}{c^2}\eta\{3a^4+2a^2b^2+3b^4-4b^2(a^2-\eta b^2)-\eta^2(a^2-b^2)(3a^2+b^2)\}$$

$$+\mathrm{Q}\frac{b^2}{c^2}\eta\{3a^4+2a^2b^2+3b^4-4b^2(b^2-\eta a^2)-\eta^2(a^2-b^2)(3a^2+b^2)\}$$

$$-\mathrm{R}\Big\{(1-\eta^2)(3a^4+2a^2b^2+3b^4)+\frac{1}{c^2}\Big(b^2(3a^4+2a^2b^2+3b^4)$$

$$-\eta(3a^6-2a^4b^2-a^2b^4-2b^6)-\eta^2b^2(3a^4+4a^2b^2+3b^4)$$

$$+\eta^3(a^2+b^2)(a^2-2b^2)(3a^2+b^2)\Big)\Big\}\Big] \quad \ldots (49).$$

$\widehat{yy}$ and $\widehat{yz}$ may be deduced from $\widehat{xx}$ and $\widehat{zx}$ respectively by interchanging P with Q, x with y, and a with b.

If P, Q, and R be of the same order of magnitude, the principal terms depending on them in the values of $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$ are likewise of the same order; but in $\widehat{zz}$, $\widehat{yz}$, and $\widehat{zx}$, the principal terms in P and Q are only of the same order as the secondary terms in R. I have thus thought it best, in (47) and (49), to retain secondary terms in the coefficients of R, and to write the second approximation value of Π. If, however, R be zero, the terms on the left-hand sides of these equations with c^2 in the denominator should be dropped.

When R alone exists, or the bodily forces are parallel to the long dimension, then, except near the ends of the long axis, $\widehat{zz}$ is large compared to $\widehat{yz}$ and $\widehat{zx}$, while these in their turn are, at most points, large compared to $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$. The hypothesis usually made in treating long rods, viz.:—

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (50),$$

is thus approximately true. $d\widehat{xx}/dx$ is, however, of the same order as $d\widehat{zx}/dz$, and $d\widehat{zx}/dx$ of the same order as $d\widehat{zz}/dz$, so that the neglect of any of the nine differential coefficients appearing in the body-stress equations would be unjustifiable.*

When the bodily forces are perpendicular to the long dimension, then—excluding special values of x, y, z—$\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$ and $\widehat{xy}$ are of the same order of magnitude, and are large compared to $\widehat{yz}$ and $\widehat{zx}$. This result differs widely from (50).

When P, Q, R are of the same order of magnitude, we may, for a rough first approximation, neglect all the stresses but $\widehat{zz}$, and take

$$\left.\begin{aligned} \widehat{zz} &= \tfrac{1}{4}\mathrm{R}\rho(c^2-z^2), \\ -s_x/\eta = -s_y/\eta = s_z &= \tfrac{1}{4}(\mathrm{R}\rho/\mathrm{E})\,(c^2-z^2), \\ \alpha/x = \beta/y &= -\tfrac{1}{4}(\mathrm{R}\rho\eta/\mathrm{E})\,(c^2-z^2), \\ \gamma &= \tfrac{1}{4}(\mathrm{R}\rho/\mathrm{E})z(c^2-\tfrac{1}{3}z^2), \\ \delta a/a = \delta b/b &= -\tfrac{1}{4}\mathrm{R}\rho\eta c^2/\mathrm{E}, \\ \delta c/c &= \tfrac{1}{6}\mathrm{R}\rho c^2/\mathrm{E} \end{aligned}\right\} \quad \ldots\ldots \quad (51).$$

The strain at any point is the same as in a long bar subjected to a tension at its ends equal per unit of section to the local value of $\widehat{zz}$.

* See Todhunter and Pearson's 'History,' vol. 2, Part II, pp. 189—191.

To a closer degree of approximation we have, for the stress system,

$$-\widehat{zx}/zx = -\widehat{yz}/yz = \widehat{zz}/(c^2-z^2) = \tfrac{1}{4}\mathrm{R}\rho \quad \ldots\ldots\ldots (52),$$

and for the strain system, in addition to s_x, s_y, and s_z, in (51),

$$\sigma_{zx}/zx = \sigma_{yz}/yz = -\tfrac{1}{4}\mathrm{R}\rho/n \quad \ldots\ldots\ldots\ldots\ldots (53),$$

where n is the rigidity.

An analysis of (52) gives, on a plane perpendicular to z, a stress $\frac{1}{4}\mathrm{R}\rho\,(c^2-z^2)$ parallel to z, and a shearing stress $-\frac{1}{4}\mathrm{R}\rho zr$ along r, the perpendicular on the axis of z; on a plane perpendicular to r a shearing stress, $-\frac{1}{4}\mathrm{R}\rho zr$, parallel to z; on any plane containing the axis of z, no stress.

Even in the general case with P, Q, R all existent, and secondary terms retained in the coefficient of R and in Π, we get for s_z the simple formula

$$s_z = \frac{\rho}{4\mathrm{E}}\Big[-\eta\,\frac{\mathrm{P}a^2+\mathrm{Q}b^2}{c^2}\,(c^2-z^2)$$
$$+\mathrm{R}\Big\{\Big(1+(1+7\eta)\,\frac{a^2+b^2}{4c^2}\Big)(c^2-z^2)-(1+2\eta)\,(x^2+y^2)$$
$$-\eta(a^2+b^2)\Big\}\Big] \ldots (54).$$

When the bodily forces are perpendicular to the long axis, the stretch parallel to that axis is thus appreciably constant over a cross section; these perpendicular forces tend to shorten or to lengthen the long axis according as they act outwards from it or towards it.

Elongated Ellipsoid Rotating about the Long Axis $2c$.

§ 11. Putting in (46) to (49)

$$\mathrm{P} = \mathrm{Q} = \omega^2, \qquad \mathrm{R} = 0,$$

we get

$$\widehat{xx} = \frac{\omega^2\rho a^2}{(1-\eta)\,(3a^4+2a^2b^2+3b^4)}\Big[\{a^4+a^2b^2+b^4-\eta\,(a^4+b^4)\}\Big(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\Big)$$
$$-(1+2\eta)\,a^2y^2\Big] \ldots (55),$$

$$\widehat{zz} = \frac{-\omega^2\rho\eta}{(1-\eta)\,(3a^4+2a^2b^2+3b^4)}\Big[\{a^4+a^2b^2+2b^4-\eta\,(a^4-b^4)\}x^2$$
$$+\{2a^4+a^2b^2+b^4+\eta\,(a^4-b^4)\}y^2$$
$$-\tfrac{1}{4}(a^2+b^2)\{(a^2+b^2)^2-\eta\,(a^2-b^2)^2\}\Big(1-\frac{z^2}{c^2}\Big)\Big] \ldots (56),$$

$$\widehat{xy} = \frac{w^2\rho\{\eta(a^2+b^2)^2 - a^4 - b^4\}xy}{(1-\eta)(3a^4+2a^2b^2+3b^4)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (57),$$

$$\widehat{zx} = \frac{w^2\rho\eta zx}{4(1-\eta^2)c^2(3a^4+2a^2b^2+3b^4)}\left[(a^2+b^2)(3a^4+2a^2b^2+3b^4)\right.$$
$$\left. -4b^2(a^4-2\eta a^2b^2+b^4) - \eta^2(a^2-b^2)(a^2+b^2)(3a^2+b^2)\right] \ldots (58).$$

$\widehat{yy}$ and $\widehat{yz}$ may be got from $\widehat{xx}$ and $\widehat{zx}$ respectively by interchanging x with y and a with b. To a first approximation $\widehat{yz}$ and $\widehat{zx}$ are negligible.

Neglecting z^2/c^2 in these formulæ, we obtain results applicable to the central portion of the long ellipsoid; these results are identical with those I have previously obtained for a long elliptic cylinder,* the axes of whose elliptic section are $2a$ and $2b$, rotating about the cylindrical axis. To deduce results for the elongated ellipsoid from those found for the infinite cylinder, we write $1-z^2/c^2$ for 1 in the constant terms in s_x, s_y, s_z, and Δ, in the coefficient of x in α and in that of y in β; and we multiply the expression for γ by $1-\frac{1}{3}z^2/c^2$.

The strain and displacement parallel to the long axis are of special interest; they are

$$s_z = -\tfrac{1}{4}\frac{w^2\rho\eta(a^2+b^2)}{\mathrm{E}}\left(1-\frac{z^2}{c^2}\right) \ldots\ldots\ldots\ldots \quad (59),$$

$$\gamma = -\tfrac{1}{4}\frac{w^2\rho\eta(a^2+b^2)}{\mathrm{E}}z\left(1-\tfrac{1}{3}\frac{z^2}{c^2}\right) \ldots\ldots\ldots\ldots \quad (60)$$

Denoting by $2l$ the length of the long elliptic cylinder, we have

$$\delta c/c = -\tfrac{1}{6}(w^2\rho\eta/\mathrm{E})(a^2+b^2) = \tfrac{2}{3}(\delta l/l) \ldots\ldots\ldots\ldots (61).$$

This enables the values of $\delta c/c$, for $\eta = 0{\cdot}25$ and $b/a = 0$, $0{\cdot}2$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, and 1, to be written down from Table XV, p. 159, of my paper on the elliptic cylinder. Table XVI of that paper gives values of $\delta a/a$ and $\delta b/b$ which apply unchanged to the long ellipsoid; while Tables XIII and XIV give its limiting angular velocity on the stress-difference and greatest strain theories.

The formula

$$\delta a/a - \delta b/b = \frac{w^2\rho(1+\eta)(a^2-b^2)\{(1+\eta)(a^2+b^2)^2+a^4+b^4\}}{3\mathrm{E}(3a^4+2a^2b^2+3b^4)} \ldots (62)$$

shows that of the principal transverse axes the longer is that which, even proportionately, is most extended.

* 'Phil. Mag.,' Aug., 1892, formulæ (80) to (83), p. 156.

Elongated Ellipsoid Rotating about a Short Axis 2a.

§ 12. We have to substitute in equations (46) to (49)

$$P = 0, \qquad Q = R = \omega^2.$$

Putting for shortness

$$4(1-\eta^2)(3a^4+2a^2b^2+3b^4) = \Pi'' \qquad \ldots\ldots\ldots\ldots (63),$$

we get

$$\widehat{xx} = -\frac{\omega^2\rho a^2}{\Pi''}\Big[\{(1-\eta^2)(a^4+3b^4)+(1-3\eta-6\eta^2)a^2b^2\}\left(1-\frac{x^2}{a^2}\right)$$
$$-\{(1-\eta^2)(4a^4+3a^2b^2+6b^4)-\eta(3+5\eta)a^2b^2\}\frac{z^2}{c^2}$$
$$-\{(1-\eta^2)(a^2+6b^2)-3\eta(3+5\eta)a^2\}y^2\Big] \ldots\ldots (64),$$

$$\widehat{yy} = \frac{\omega^2\rho b^2}{\Pi''}\Big[\{5a^4+3a^2b^2+3b^4+3\eta a^2b^2-\eta^2(5a^4-2a^2b^2+3b^4)\}\left(1-\frac{y^2}{b^2}\right)$$
$$-a^2\{(1+3\eta+4\eta^2)b^2+2(1-\eta^2)a^2\}\left(3\frac{x^2}{a^2}+\frac{z^2}{c^2}\right)\Big] \ldots\ldots (65),$$

$$\widehat{zz} = \frac{\omega^2\rho c^2}{\Pi''}\Big[-\{3a^4+2a^2b^2+3b^4+\eta(5a^4+9a^2b^2+6b^4)-\eta^2(3a^4-a^2b^2-6b^4)$$
$$-\eta^3(a^2-b^2)(5a^2+9b^2)\}\frac{x^2}{c^2}$$
$$-\{3a^4+2a^2b^2+3b^4+\eta(10a^4+a^2b^2+9b^4)+\eta^2(6a^4+a^2b^2-3b^4)$$
$$+\eta^3(a^2-b^2)(5a^2+9b^2)\}\,y^2/c^2$$
$$+\{\tfrac{1}{4}\Pi''+\text{terms of order } a^2/c^2 \text{ and } b^2/c^2\}\left(1-\frac{z^2}{c^2}\right)\Big] \ldots\ldots (66),$$

$$\widehat{yz} = -\tfrac{1}{4}\omega^2\rho yz \qquad \ldots\ldots\ldots\ldots (67),$$

$$\widehat{zx} = -\tfrac{1}{4}\omega^2\rho zx \qquad \ldots\ldots\ldots\ldots (68),$$

$$\widehat{xy} = \frac{\omega^2\rho xy}{\Pi''}\{(1-\eta^2)(a^4-3b^4)+2\eta(3+5\eta)a^2b^2\} \ldots\ldots\ldots\ldots (69).$$

First approximations to the values of the stresses, strains, displacements, and increments of the semi-axes may be obtained by writing ω^2 for R in (51).

To this degree of approximation it makes no difference which of the two short axes is that about which rotation takes place. The two short axes shorten to the same extent per unit of length, while the long axis lengthens.

By writing ω^2 for Q and R in (54), we get a very close approximation to the strain s_z parallel to the long axis.

Application of Method of Mean Values.

§ 13. Let $t_1, = 4a/3, t_2, t_3$ be the mean lengths of material lines parallel to the principal axes $2a, 2b, 2c$, and let v be the volume of the ellipsoid. Then it is easily proved from the results in my paper* on the mean values of strains, &c., that the elastic increments in the general case are given by

$$\delta t_1/t_1 = \tfrac{1}{5}(\rho/\mathrm{E})\{\mathrm{P}a^2 - \eta(\mathrm{Q}b^2 + \mathrm{R}c^2)\} \quad \ldots\ldots\ldots (70),$$

$$\delta v/v = \tfrac{1}{15}(\rho/k)(\mathrm{P}a^2 + \mathrm{Q}b^2 + \mathrm{R}c^2) \quad \ldots\ldots\ldots\ldots (71),$$

$$\delta t_1/t_1 + \delta t_2/t_2 + \delta t_3/t_3 = \delta v/v \quad \ldots\ldots\ldots\ldots\ldots\ldots (72).$$

In (71) k denotes the "bulk modulus."

For the gravitating nearly spherical ellipsoid, (70) gives

$$\delta t_1/t_1 = -\tfrac{4}{15}\pi\mu\rho^2 a^2 \frac{1-2\eta}{\mathrm{E}}\left\{1 - \tfrac{1}{5}\frac{1-4\eta}{1-2\eta}\frac{2a^2 - b^2 - c^2}{a^2}\right\} \quad \ldots (73).$$

Comparing (73) and (19) we see that in a gravitating perfect sphere

$$\delta t_1/t_1 = \delta a/a,$$

or the reductions per unit length of a diameter and of the mean parallel chord are identical.

For the increment in volume in the gravitating nearly spherical ellipsoid, (71) gives

$$\delta v/v = -4\pi\mu\rho^2(a^2 + b^2 + c^2)/(45k) \quad \ldots\ldots\ldots\ldots (74).\dagger$$

For the gravitating very flat spheroid of § 7, we find

$$\left.\begin{aligned} \delta t_1/t_1 &= -\pi^2\mu\rho^2 ac(1-\eta)/5\mathrm{E}, \\ \delta t_3/t_3 &= 2\pi^2\mu\rho^2 ac\eta/5\mathrm{E}, \\ \delta v/v &= -2\pi^2\mu\rho^2 ac(1-2\eta)/5\mathrm{E} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (75).$$

For the general case of an ellipsoid rotating about $2a$, we have

$$\left.\begin{aligned} \delta t_1/t_1 &= -\tfrac{1}{5}\frac{\omega^2\rho\eta}{\mathrm{E}}(b^2 + c^2), \\ \delta t_2/t_2 &= \tfrac{1}{5}\frac{\omega^2\rho}{\mathrm{E}}(b^2 - \eta c^2), \\ \delta v/v &= \omega^2\rho(b^2 + c^2)/(15k) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (76).$$

* 'Camb. Phil. Soc. Trans.,' vol. 15, pp. 313—337.

† Agrees with formula (105), p. 335, 'Camb. Trans.,' *loc. cit.*

A material line parallel to an axis $2b$, perpendicular to the axis of rotation, is exposed to two opposing actions. The components of "centrifugal" force in its own direction tend to lengthen it, while those in the perpendicular direction $2c$ tend to shorten it. On an average the former action will prevail so long as the mean dimension parallel to $2b$ bears to that parallel to $2c$ a ratio exceeding $\sqrt{\eta} : 1$.

Approximate Methods.

§ 14. Suppose a body symmetrical with respect to the coordinate planes, of great length, $2c$, in the direction of z, to be acted on by the bodily forces whose components are $\mathrm{P}x$, $\mathrm{Q}y$, $\mathrm{R}z$. The stresses over the plane $z = z$ must balance the force whose components are

$$\int_z^c\iint \rho \, \mathrm{P}x \, dx \, dy \, dz, \text{ \&c.}$$

Since every section perpendicular to z has its c.g. on that axis, it is clear the integrals vanish, which give the components parallel to x and y. If now we assume that when there is no rapid variation in the cross section, σ, as z alters, $\widehat{zz}$ is large compared to the other stresses, and is approximately uniform over σ, then we may take as a first approximation

$$\widehat{zz} = \frac{\mathrm{R}\rho}{\sigma}\int_z^c \iint z \, dz \, dy \, dx.$$

For an elongated ellipsoid this gives, as in (51),

$$\widehat{zz} = \tfrac{1}{4}\mathrm{R}\rho\,(c^2-z^2).$$

Taking this as the sole stress, we obtain from the ordinary stress-strain relations the same values of s_x, s_y, s_z as in (51), and thence by integration the correct first approximation values of the displacements.

For a thin elliptic disc in which one of the axes, $2c$, of the elliptic section is very large compared to the other or to the thickness, we find as first approximations

$$\left.\begin{aligned} \widehat{zz} &= \tfrac{1}{3}\mathrm{R}\rho\,(c^2-z^2), \\ \alpha/x = \beta/y &= -\tfrac{1}{3}\mathrm{R}\rho\eta\,(c^2-z^2)/\mathrm{E}, \\ \gamma &= \tfrac{1}{3}\mathrm{R}\rho z(c^2-\tfrac{1}{3}z^2)/\mathrm{E} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (77).$$

If, for instance, the disc be rotating about its thickness, *i.e.*, the axis of the cylinder, then $\mathrm{R} = \omega^2$ and

$$\left.\begin{aligned} \alpha/x = \beta/y &= -\tfrac{1}{3}\omega^2\rho\eta\,(c^2-z^2)/\mathrm{E}, \\ \gamma &= \tfrac{1}{3}\omega^2\rho z\,(c^2-\tfrac{1}{3}z^2)/\mathrm{E} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (78).$$

As a third example, suppose the section perpendicular to the long axis uniform in shape and area, we then get

$$\left.\begin{aligned} \widehat{zz} &= \tfrac{1}{2}\mathrm{R}\rho\,(c^2-z^2), \\ \alpha/x = \beta/y &= -\tfrac{1}{2}\,\mathrm{R}\rho\eta\,(c^2-z^2)/\mathrm{E}, \\ \gamma &= \tfrac{1}{2}\mathrm{R}\rho z\,(c^2-\tfrac{1}{3}z^2)/\mathrm{E} \end{aligned}\right\} \ldots\ldots\ldots\ldots (79).$$

A special instance of this last case is presented by an elongated cylinder rotating about a perpendicular to its length through its centre. Putting $\mathrm{R} = \omega^2$, we have

$$\left.\begin{aligned} \alpha/x = \beta/y &= -\tfrac{1}{2}\,\omega^2\rho\eta\,(c^2-z^2)/\mathrm{E}, \\ \gamma &= \tfrac{1}{2}\,\omega^2\rho z\,(c^2-\tfrac{1}{3}z^2)/\mathrm{E} \end{aligned}\right\} \ldots\ldots\ldots\ldots (80).*$$

Comparing the several cases of rotation, we have an interesting illustration of how the effects of the "centrifugal" force increase as the mean distance of the substance of the solid from the axis of rotation becomes larger. If the limiting angular velocity permissible in the elongated ellipsoid rotating about a short axis be taken as 100, then the limiting angular velocities in the thin elliptic disc and the elongated cylinder rotating about their short axes—the material, and the length of the long dimension being the same for all—are approximately 87 and 71 respectively, both on the stress-difference and greatest strain theories. A caution must, however, be added that in bodies of such elongated form rotating about a short axis, a sudden change in the angular velocity may prove disastrous.

"Micro-Metallography of Iron. Part I." By THOMAS ANDREWS, F.R.S., M.Inst.C.E. Received December 15, 1894,—Read January 24, 1895.

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* This is more exact for a long beam of rectangular cross section than the result I obtained in the 'Quarterly Journal' for 1888, p. 29.